A new topographic functional

Edward Earl and David Metzler

1 Introduction

The simplest and most familiar number associated to a mountain peak is the elevation of its summit above sea level. However, absolute elevation often does not correlate well with the visual impressiveness of a peak, which has more to do with the amount of local relief and the steepness of the flanks of the peak. For example, the summit of Mount Elbert, the highest point in the Rocky Mountains, is 4401 meters above sea level[1], while Devils Thumb[1], a striking rock spire on the border between Alaska and British Columbia, rises only to 2767 meters[2]. Based on pure elevation, Elbert far surpasses Devils Thumb. However, Mount Elbert rises from a high base in central Colorado, so its local relief is not nearly as great as its elevation would indicate; nor is it a particularly steep peak. For example, Elbert rises about 1600 meters (one mile) over a horizontal distance of 6.5 kilometers on its southeast flank[2]—which is not unimpressive. However, the northwest face of Devils Thumb soars an amazing 2000 meters in 1.6 km, and it is similarly steep in other directions. To get 2000 meters of vertical relief from the summit of Mount Elbert, one has to go about 30 km away, to the town of Aspen; if one goes 30 km from Devils Thumb, one gets to tidewater, yielding 2767 meters of relief. See Table 1 for representative profiles of the two peaks.

In this article we introduce a functional that takes into account the relief and steepness of a peak in a mathematically elegant way, and which has substantial correlation with the visual impressiveness of the peak. In fact, our functional can be applied to any point on a landscape (not necessarily a summit—for example, see the discussion below of the famous granite cliff of El Capitan in Yosemite), or indeed, any point on the graph of a function. We will also briefly introduce two concepts derived from the main functional; one takes into account how independent a particular feature is from nearby "better" features, and the other calculates a kind of "ruggedness" for a domain.

A pedagogical note: using the basic definitions provides good exercises in multivariable calculus, suitable for strong students in an introductory course. Proving theorems about these measures involve good workouts with elementary real and functional analysis.

[1] There is no apostrophe in the official spelling of the name of this peak.

[2] One can verify these numbers using the public-domain mapping website mapper.acme.com, among others.

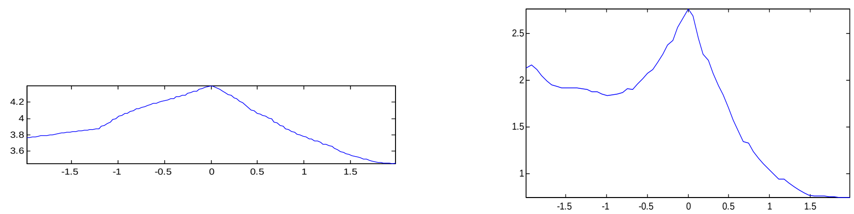


Table 1: Mount Elbert (left) and Devils Thumb (right) profiles

2 Omnidirectional Relief and Steepness (ORS)

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a bounded, Lebesgue measurable function, thought of as the height function of a landscape. (We do not require h to be continuous, to permit the presence of vertical cliffs.[3]) Consider a fixed *base point* $\mathbf{p} \in \mathbb{R}^2$, and a corresponding *reference point* $(\mathbf{p}, h_0)$. (It is theoretically useful, and no more complicated, to let the height h_0 of the reference point vary independently, so h_0 need not equal $h(\mathbf{p})$. Physically, one can imagine, for example, $h_0 > h(\mathbf{p})$ to be the height of the top of a flagpole placed atop a peak. However we will primarily be interested in the case where $h_0 = h(\mathbf{p})$.) We will define a functional of this data, which we call *omnidirectional relief and steepness* (ORS), which will capture a kind of average of the relief and steepness of the terrain as viewed from the reference point.

More precisely, let $h \in L^\infty\left(\mathbb{R}^2\right)$, and let $(\mathbf{p}, h_0) \in \mathbb{R}^2 \times \mathbb{R}$. We will presently define ORS of the reference point $(\mathbf{p}, h_0)$ relative to the landscape h, yielding a functional

$$\begin{aligned} \mathrm{ORS} : \mathbb{R}^2 \times \mathbb{R} \times L^\infty\left(\mathbb{R}^2\right) &\to \mathbb{R} \\ (\mathbf{p}, h_0; h) &\mapsto \mathrm{ORS}\left(\mathbf{p}, h_0; h\right) \end{aligned}$$

(In fact we will define a whole family of possible functionals, but we will immediately specialize to one particularly appealing case.)

We first consider a simple landscape, both to fix ideas and to define an important normalization for the general case.

Definition 1 *Let $h_0, b > 0$, let $s = h_0/b$, and let $\phi = \arctan s$. Then the* ***cone***

[3] We could use S^2 as the domain, to take into account the spherical nature of the Earth, but we will see that all of the calculations localize strongly, making the difference minuscule. Generalizing everything in this paper to $\mathbb{R}^n$ is straightforward, but we use $\mathbb{R}^2$ throughout for simplicity and because of the application to physical landscapes. However, we do not take into account overhanging cliffs, since that would vastly complicate the mathematical model.

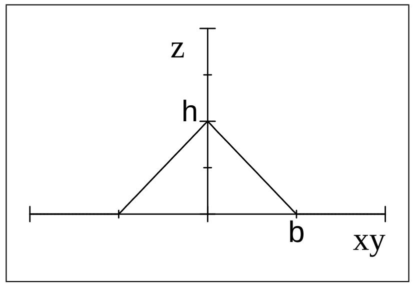


Figure 1: Cross-section of cone function with height 1 and slope 1

function *c associated to h_0, b is (in polar coordinates)*

$$c(\mathbf{x}) = \begin{cases} h_0 - sr, & r < b \\ 0 & r > b \end{cases}$$

(We suppress the dependence on h_0, b for tidiness.) Note that s is the slope of the cone, and ϕ is the angle its sides make with the xy-plane.

See Figure 1 for the cross-section of the cone. We wish to define the ORS of the summit of this cone, i.e. ORS $(\mathbf{0}, h_0; c)$. It should take into account its height, and also its steepness. The combination $h_0 s$ does not work, since it is unbounded for large s, even if h_0 is small. The combination $h_0\phi$ is just as natural, and is bounded. We will actually choose $\frac{2}{\pi}h_0\phi$ ("height times angle over 90°"), so that the limiting case $\phi \to \pi/2$, which we will call a *flagpole*, yields simply h_0. Hence we have the following.

Definition 2 *We say that ORS is* ***angle-normalized*** *if it yields $\frac{2}{\pi}h_0\phi$ when applied to the vertex of the cone with height h and angle ϕ:*[4]

$$\text{ORS}\,(\mathbf{0}, h_0; c) = \frac{2}{\pi}h_0\phi$$

Note two further important features of the conical case: first, if two cone functions c_1, c_2 share the same angle but have different heights $h_2 = Ah_1$, then the ORS of c_2 will be A times the ORS of c_1. In other words, scaling up every dimension (heights and horizontal distances) by a factor of A results in scaling up ORS by the same factor. We will see below that this homogeneity, or scale-covariance, property is true of ORS in general; in particular, it means that ORS has meaningful units, namely, units of length. (In the topographic examples below, ORS is given in meters.)

[4] We discuss other possible normalizations just after Theorem 5.

Second, if we take a low-slope cone c with height h and base $b >> h$ and scale up h by a factor of A, leaving b unchanged, then the ORS will increase by approximately A^2. This low-slope quadratic behavior is also a general feature of ORS.

Now we turn to the general case of a non-conical peak or other topographic feature. We imagine standing at the reference point—say the summit of a mountain—and looking down in all directions, gauging the impressiveness of the view. We want to take some sort of average of the impressiveness information obtained by looking in all directions. One can also think of stationing a host of tourists (mathematically, these will be called *sample points*) everywhere around the mountain, all looking up at the summit, and surveying them for their idea of the impressiveness of the summit.[5] Hence ORS will involve an integral over the set of all sample points; we will denote a typical sample point by $\mathbf{x}$, and we will set $r = \|\mathbf{p} - \mathbf{x}\|$, the distance from the reference point to a sample point.

For every sample point $\mathbf{x}$, we calculate the slope $u(\mathbf{x}) = (h_0 - h(\mathbf{x}))/r$. If we integrated u itself, the integral over all $\mathbf{x} \in \mathbb{R}^2$ would clearly diverge for most landscapes. Instead, we use an appropriate function to turn u into a sensible integrand. We first present a general definition, using an arbitrary such function, and then use the cone normalization to determine what function we desire.

Definition 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(u) = 0$ for $u \leq 0$.*[6] *Let $h \in L^\infty(\mathbb{R}^2)$, and let $(\mathbf{p}, h_0) \in \mathbb{R}^2 \times \mathbb{R}$. Let $r = \|\mathbf{x} - \mathbf{p}\|$ be the radial coordinate based at $\mathbf{p}$, and let $u(\mathbf{x}) = (h_0 - h(\mathbf{x}))/r$. The* ***omnidirectional relief and steepness (ORS)*** *of the reference point $(\mathbf{p}, h_0)$ relative to the landscape h, using f, is*

$$\begin{aligned} \mathrm{ORS}_f(\mathbf{p}, h_0; h) &= \|f \circ u\|_2 \\ &= \left[\iint_{\mathbb{R}^2} f^2 \left(\frac{h_0 - h(\mathbf{x})}{r} \right) dA(\mathbf{x}) \right]^{1/2} \end{aligned}$$

Before examining the general properties of ORS, we first derive the correct function f based on our normalization.

Proposition 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $f(u) = o(u^{1+\varepsilon})$ for some $\varepsilon > 0$, as $u \to 0$. Let $h_0, b > 0$ and let c be the associated cone function, with slope $s = h_0/b$. Then $\mathrm{ORS}_f(0, 0, h_0; c) = h_0 F(s)$, where F satisfies the initial value problem*

$$\frac{1}{\pi} \left(F^2(s)\right)' = \frac{1}{s^2} \left(f^2(s)\right)', \qquad F(0) = 0.$$

[5] Note that ORS ignores line-of-sight issues: we make no distinction between points that are actually in view from the reference point and points that are obscured by intervening terrain. Hence phrases such as "looking up at the mountain" should not be taken too literally.

[6] It is not absolutely necessary to require that f vanish for negative u. It has the effect of ignoring surrounding higher terrain in evaluating the reference point. This usually has a negligible effect when the reference point is a summit, which is our main application. Dropping this requirement turns out to make the reduced version of ORS, discussed at the end of this paper, difficult to define.

Proof. Defining F as in the theorem, we have

$$\begin{aligned}
h_0 F(s) &= \left(\iint\limits_{\mathbb{R}^2} f^2 \left(\frac{h_0 - c(r)}{r} \right) dA \right)^{1/2} \\
h_0^2 F^2(s) &= 2\pi \int_0^\infty f^2 \left(\frac{h_0 - c(r)}{r} \right) r \, dr. \\
\frac{1}{2\pi} h_0^2 F^2(s) &= \int_0^b f^2(s)\, r \, dr + \int_b^\infty f^2 \left(\frac{h_0}{r} \right) r \, dr \\
&= \frac{1}{2} f^2(s)\, b^2 + \int_0^s f^2(u) \frac{h_0^2}{u^3} \, du
\end{aligned}$$

where we have set $u = h_0/r$ and hence $du = -\left(h_0/r^2\right) dr$ or $r\, dr = -\left(r^3/h_0\right)\, du = -\left(h_0^2/u^3\right)\, du$. Note that the order of vanishing assumed for f makes all the integrals converge. Hence

$$\begin{aligned}
\frac{1}{2\pi} h_0^2 F^2(s) &= \frac{1}{2} f^2(s) \frac{h_0^2}{s^2} + \int_0^s f^2(u) \frac{h_0^2}{u^3} \, du \\
\frac{1}{2\pi} F^2(s) &= \frac{1}{2s^2} f^2(s) + \int_0^s f^2(u) \frac{du}{u^3}
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\frac{1}{2\pi} F^2(s) &= \frac{1}{2s^2} f^2(s) - \frac{1}{2s^2} f^2(s) \\
&\quad + \lim_{u \to 0} \frac{f^2(u)}{2u^2} + \frac{1}{2} \int_0^s \left(f^2(u)\right)' \frac{du}{u^2} \\
&= \frac{1}{2} \int_0^s \left(f^2(u)\right)' \frac{du}{u^2}
\end{aligned}$$

or, taking the derivative of both sides,

$$\frac{1}{\pi} \left(F^2(s)\right)' = \frac{1}{s^2} \left(f^2(s)\right)', \qquad F(0) = f(0) = 0.$$

■

Proposition 5 *Let*

$$f(u) = \left[\frac{4}{\pi^3} \left(2u \arctan u - \ln\left(u^2 + 1\right) - \arctan^2 u \right) \right]^{1/2} \tag{1}$$

for $u \geq 0$ and $f(u) = 0$ for $u < 0$. Then the function F associated to f by Theorem 4 is $F(s) = \frac{2}{\pi} \arctan s$, and hence the resulting ORS_f *is angle-normalized:*

$$\mathrm{ORS}_f(0, 0, h_0; c) = \frac{2}{\pi} h_0 \phi.$$

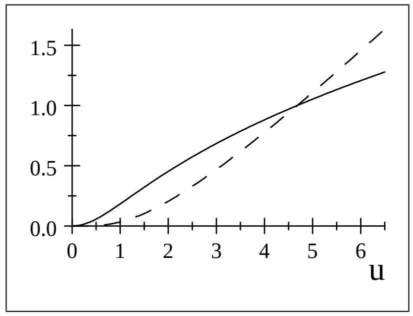


Figure 2: Plot of f (solid) and f^2 (dashed)

Proof.

$$
\begin{aligned}
f^2(s) &= \frac{1}{\pi}\int_0^s u^2\left(F^2(u)\right)'\,du \\
&= \frac{4}{\pi^3}\int_0^s u^2\left(\arctan^2 u\right)'\,du \\
&= \frac{8}{\pi^3}\int_0^s \frac{u^2}{u^2+1}\arctan u\,du \\
&= \frac{8}{\pi^3}\int_0^s \left(1-\frac{1}{u^2+1}\right)\arctan u\,du \\
&= \frac{4}{\pi^3}\left[2\int_0^s \arctan u\,du - \int_0^s \left(\arctan^2 u\right)'\,du\right] \\
&= \frac{4}{\pi^3}\left[2u\arctan u\big|_0^s - 2\int_0^s \frac{u}{u^2+1}\,du - \arctan^2 s\right] \\
&= \frac{4}{\pi^3}\left[2s\arctan s - \ln\left(s^2+1\right) - \arctan^2 s\right].
\end{aligned}
$$

■

We exclusively use this angle-normalized f, shown in Figure 2, in our calculations of ORS in this paper. However we can say a word about what happens when one chooses different functions for f. Since ORS combines information about local relief with information about steepness, there is an issue of how much to weight relief versus steepness: should we assign a greater value to a very steep, but only moderately high peak, or to a moderately steep, but very high peak? At the risk of making apples-to-oranges comparisons, we boldly proceed to assign one number that makes a certain tradeoff between relief and steepness. Different choices for f will result in somewhat different tradeoffs, either more "heightist" (favoring relief over steepness) or more "slopist" (the opposite). In past work we have also tried other normalizations, notably $F(s) = s/(s+1)$, which is more heightist than the angle normalization. We work with angle normalization for reasons of simplicity, elegance, and a good fit with visual impressiveness.

For the remainder of this paper, we will use the modified slope integrand f given in Theorem 5, and we will suppress f from the notation; that is, we define

$$\mathrm{ORS}(\mathbf{p}, h_0; h) = \mathrm{ORS}_f(\mathbf{p}, h_0; h)$$

With this definition, ORS has many good properties, including strong versions of continuity, which are essential for dealing with the discretized data encountered in practice.[7] Note that the square root in the definition is an order-preserving function; hence for the purposes of comparing peaks (one of our main uses for ORS), it is enough to use ORS^2, which will be simpler to analyze. One can think of the square root serving mainly to make normalization easier (in particular, it produces a quantity with units of length). The root does make it tricky to analyze the behavior of ORS for landscapes where ORS is very small. This is not a major concern for our purposes, since we focus primarily on reference points for which ORS is relatively large. Also, taking the square root halves relative error, so any relative error result for ORS^2 yields a corresponding, and stronger, relative error result for ORS.

Proposition 6 *The functional* $\mathrm{ORS} : \mathbb{R}^2 \times \mathbb{R} \times L^\infty\left(\mathbb{R}^2\right) \to \mathbb{R}$ *has the following properties:*

1. ORS *is weakly increasing as a function of* h_0 *and weakly decreasing as a function of* h*: for every* $\mathbf{p} \in \mathbb{R}^2$*,* $h_0, k_0 \in \mathbb{R}$*, and* $h, k \in L^\infty$*, if* $h_0 \leq k_0$ *and* $h \geq k$*, then* $\mathrm{ORS}(\mathbf{p}, h_0; h) \leq \mathrm{ORS}(\mathbf{p}, k_0; k)$*.*

2. ORS *is bounded by the maximum height of the landscape: for every* $\mathbf{p} \in \mathbb{R}^2$*,* $h_0 \in \mathbb{R}$*, and* $h \in L^\infty$*,* $\mathrm{ORS}(\mathbf{p}, h_0; h) \leq \|h_0 - h\|_\infty$*. In particular, it is finite for any bounded landscape and any reference point.*

3. ORS *is invariant under vertical translation and horizontal translation: for every* $\mathbf{p} \in \mathbb{R}^2$*,* $h_0 \in \mathbb{R}$*,* $h \in L^\infty$*,* $a \in \mathbb{R}$*, and* $\mathbf{q} \in \mathbb{R}^2$*,*

$$\begin{aligned} \mathrm{ORS}(\mathbf{p}, h_0 + a; h + a) &= \mathrm{ORS}(\mathbf{p}, h_0; h) \\ \mathrm{ORS}(\mathbf{p} + \mathbf{q}, h_0; h(\mathbf{x} - \mathbf{q}) &= \mathrm{ORS}(\mathbf{p}, h_0; h) \end{aligned}$$

4. ORS *is invariant under reflections and rotations about the reference point: let* A *be a 2 by 2 orthogonal matrix and define* $h_A(\mathbf{x}) = h(A(\mathbf{x} - \mathbf{p}) + \mathbf{p})$*. Then*

$$\mathrm{ORS}(\mathbf{p}, h_0; h_A) = \mathrm{ORS}(\mathbf{p}, h_0; h)$$

5. ORS_f *is scale-covariant (if we scale the landscape both horizontally and vertically), and in particular it has units of length. That is, if* h_M *is obtained from* h *by dilating horizontally about the point* $\mathbf{p}$ *by* $M > 0$ *and scaling vertically by* M *(i.e.* $h_M(\mathbf{x}) = M \cdot h\left((\mathbf{x} - \mathbf{p})/M + \mathbf{p}\right)$*) then*

$$\mathrm{ORS}(\mathbf{p}, M h_0; h_M) = M \cdot \mathrm{ORS}(\mathbf{p}, h_0; h)$$

[7] In particular, the Lipschitz continuity in Corollary 13 would not hold if we used a 1-norm instead of a 2-norm, which might otherwise seem simpler.

Proof. Monotonicity (1) and vertical and horizontal translation invariance (3) are clear from the definition. Invariance under reflections and rotations follows from the corresponding invariance of the integral. Scale-covariance follows from the change of variables indicated in item 5. The bound given in item 2 follows from monotonicity and the flagpole case of the cone normalization; hence we will refer to this bound as the flagpole bound. ■

Remark 7 *By using vertical and horizontal invariance, we can always reduce to the case where the reference point is the origin and the reference height is zero. We do this below for simplicity, denoting the result by* $\mathrm{ORS}(h) = \mathrm{ORS}(\mathbf{0}, 0; h)$*. Note that in any statement involving a variation of the landscape* h*, we can recover a more general version, with variation in* h_0 *as well: for example, simply replace any quantity of the form* $\|h-k\|_\infty$ *by* $\|(h_0 - h) - (k_0 - k)\|$*.*

Before turning to results about the continuity and robustness of ORS, we need a lemma about the function f which appears in the definition. This lemma summarizes all of the features of f that are necessary for the results about ORS that follow.

Lemma 8 *The function* f *defined in Proposition 5 is* C^1 *on* $\mathbb{R}$ *and has the following properties for* $u > 0$*. (Recall that* f *is identically zero for* $u \leq 0$*.)*

1. f *is strictly increasing.*
2. $f^2(u) = \frac{2}{\pi^3}u^4 + O(u^6)$*.*
3. $f(u) = \sqrt{\frac{2}{\pi^3}} \cdot u^2 + O(u^3)$ *as* $u \to 0^+$ *and* $f(u) \leq \min\left\{\sqrt{\frac{2}{\pi^3}} \cdot u^2, \frac{2}{\pi}\sqrt{u}\right\}$*.*
4. f^2 *is strictly convex.*
5. $0 \leq \left(f^2\right)'(u) < \frac{4}{\pi^2}$ *and* $\left(f^2\right)'(u) \leq \frac{8}{\pi^3}u^3$*.*

Proof. Let $u > 0$. The function

$$f^2(u) = \frac{4}{\pi^3}\left(2u \arctan u - \ln\left(u^2+1\right) - \arctan^2 u\right)$$

is clearly C^∞. Its Taylor expansion at $u = 0$ is

$$\begin{aligned} f^2(u) &= \frac{4}{\pi^3}\left(2u\left(u - \frac{u^3}{3}\right) - \left(u^2 - \frac{u^4}{2}\right) - \left(u - \frac{u^3}{3}\right)^2\right) + O(u^6) \\ &= \frac{2}{\pi^3}u^4 + O(u^6) \end{aligned}$$

Hence

$$\begin{aligned} f(u) &= \sqrt{\frac{2}{\pi^3}u^4\left(1 + O\left(u^2\right)\right)} \\ &= \sqrt{\frac{2}{\pi^3}}u^2\left(1 + O(u^2)\right) \end{aligned}$$

This shows that, even with the proviso that $f(u) = 0$ for $u < 0$, f is C^1 for all $u \in \mathbb{R}$.

Next we calculate the derivative of the squared function:

$$\left(f^2\right)'(u) = \frac{8}{\pi^3} u^2 \frac{\arctan u}{u^2+1}$$

(recall from Prop.4 that it is not accidental that this is relatively simple). This is clearly positive for $u > 0$; hence f^2 and f are both increasing (in fact, strictly increasing as long as $u > 0$). (Again, this follows also from Prop. 4.) Since $\arctan u < \min\left\{\frac{\pi}{2}, u\right\}$ for $u > 0$, we see also that $\left(f^2\right)'(u) < \min\left\{\frac{4}{\pi^2}, \frac{8}{\pi^3}u^3\right\}$ for $u > 0$. We take the second derivative and obtain

$$0 < \left(f^2\right)''(u) = \frac{8}{\pi^3} \frac{u\,(u + 2\arctan u)}{\left(u^2+1\right)^2} < \frac{24}{\pi^3} u^2 = \frac{d^2}{du^2}\left(\frac{2}{\pi^3}u^4\right) \qquad (u > 0)$$

which shows that $\left(f^2\right)$ is convex, and also, since $f^2(0) = \left(f^2\right)'(0) = 0$, that

$$f^2(u) < \frac{2}{\pi^3} u^4$$

and hence that

$$f(u) < \sqrt{\frac{2}{\pi^3}} u^2$$

as desired. ■

We now consider the sensitivity of ORS and ORS^2 to the landscape data h (and hence also to the height h_0 of the reference point, as in Remark 7). We certainly want continuity, but we actually want a bit more; continuous functions can have unpleasantly large derivatives. This is important when dealing with discrete, and often somewhat inaccurate, digital data. In fact, a previous attempt at defining such a function using a 1-norm instead of a 2-norm led to poor behavior in this regard.

To quantify the sensitivity of $\mathrm{ORS}^2(h)$ to variations in h, we recall the following standard notion from functional analysis.[3]

Definition 9 *Given a function $F : V \to W$ between two topological vector spaces, the Gâteaux differential of F is the function dF given by*

$$dF(h, v) = \left.\frac{d}{dt}\right|_{t=0} F(h + tv)$$

F is said to be Gâteaux differentiable at $h \in V$ if dF exists for all $v \in V$.

In general, dF need not be continuous or linear. In our case, we are most interested in the following. Suppose that V and W are Banach spaces, and that F is Gâteaux differentiable at h. Then define

$$mF(h) = \sup_{\|v\|=1} \|dF(h, v)\|$$

(which may be infinite). If F is actually (Fréchet) differentiable at h, this is clearly just the norm of the derivative, as a linear operator between Banach spaces. It measures the worst-case sensitivity of F at h. We are interested in a simpler quantity, namely

$$MF(H) = \sup_{\|h\|=H} mF(h) = \sup_{\|h\|=H} \sup_{\|v\|=1} \|dF(h,v)\|$$

which gives the worst-case sensitivity of F over all inputs of given norm H. We are interested in the case where $V = L^\infty$ and $W = \mathbb{R}$. For example, if $F(h) = \|h\|_\infty^2$, a simple calculation yields $MF(H) = 2H$. With this notation, we can state our main result about the sensitivity of ORS.

Theorem 10 *The worst-case sensitivity of* ORS^2 *satisfies*

$$M\mathrm{ORS}^2(H) = 2H$$

That is, it is exactly as sensitive, in the worst case, as the function H^2. For ORS *itself, we have*

$$m\mathrm{ORS}(h) \leq \frac{\|h\|_\infty}{\mathrm{ORS}(h)}$$

Before proving the theorem, we first note the following easy consequence of monotonicity, whose proof we omit.

Lemma 11 *Let $H > 0$ be fixed and consider all pairs of landscapes $h, k \in L^\infty$, with $\|h-k\|_\infty = H$. Then $\left|\mathrm{ORS}^2(h) - \mathrm{ORS}^2(k)\right|$ is maximized when $h-k$ is a constant function (a.e.).*

Proof of the Theorem. Let $h \in L^\infty$. By the lemma, to calculate $m\mathrm{ORS}(h)$, we need only consider the case where v is constant function; let's say $v = z$ everywhere. So

$$\begin{aligned}
m\mathrm{ORS}(h) &= \left.\frac{d}{dz}\right|_{z=0} \mathrm{ORS}_f^2(h+z) \\
&= \left.\frac{d}{dz}\right|_{z=0} \int_{\mathbb{R}^2} f^2\left(\frac{h(\mathbf{x})+z}{r}\right)\, dA \\
&= \int_{\mathbb{R}^2} \left.\frac{\partial}{\partial z}\right|_{z=0} \left(f^2\left(\frac{h(\mathbf{x})+z}{r}\right)\right)\, dA \\
&= \int_{\mathbb{R}^2} \left(f^2\right)'\left(\frac{h(\mathbf{x})}{r}\right)\frac{1}{r}\, dA
\end{aligned}$$

where we can pass the derivative inside the integral since $\left(f^2\right)'\left(\frac{h(\mathbf{x})}{r}\right)\frac{1}{r}$ is integrable.[4] (It is integrable near the origin since $\left(f^2\right)'$ is bounded, and at infinity since $\left(f^2\right)'(u) \leq \frac{8}{\pi^3}u^3$, both by Lemma 8.) Now let $H > 0$ and consider all functions h with $\|h\|_\infty = H$. Since $\left(f^2\right)'$ is increasing by Lemma 8, $m\mathrm{ORS}(h)$ will be maximized when h is a constant function, with value H. But this reduces us to the case where $h = H$ and v are both constant, that is, the flagpole case, and this is normalized to give

$$\mathrm{ORS}^2(H) = H^2$$

for which we have already noted that

$$M\mathrm{ORS}^2(H) = 2H$$

The result about ORS itself follows by the chain rule:

$$\begin{aligned} m\mathrm{ORS}(h) &= \left.\frac{d}{dz}\right|_{z=0} \sqrt{\mathrm{ORS}^2(h+z)} \\ &= \frac{\left.\frac{d}{dz}\right|_{z=0} \mathrm{ORS}^2(h+z)}{2\sqrt{\mathrm{ORS}^2(h)}} \\ &\leq \frac{M\mathrm{ORS}^2(\|h\|_\infty)}{2\sqrt{\mathrm{ORS}^2(h)}} \\ &= \frac{\|h\|_\infty}{\sqrt{\mathrm{ORS}^2(h)}} \end{aligned}$$

■

Corollary 12 *The function* ORS^2 *is locally Lipschitz continuous, and the Lipschitz bound depends only on* $\|h\|_\infty$. *More precisely, on any set* S *with* $\|h\|_\infty \leq H$ *for all* $h \in S$,

$$\left|\mathrm{ORS}^2(h_0) - \mathrm{ORS}^2(h_1)\right| \leq 2H$$

for all h_0, h_1 *in* S.

Proof. For $h_0, h_1 \in S$, let $h_t = th_1 + (1-t)h_0$. The corollary follows from the mean value theorem applied to the function $t \mapsto \mathrm{ORS}^2(h_t)$, since the preceding theorem implies that the derivative of this function is bounded by $2H$. ■

Corollary 13 *The function* ORS *is continuous, and it is locally Lipschitz continuous away from the zero (a.e.) landscape. Further, on a set* S *on which* ORS *is bounded away from zero,* ORS *is uniformly Lipschitz continuous.*

Proof. ORS is continuous since ORS^2 is. If h is not the zero landscape, then $\mathrm{ORS}(h) \neq 0$, and by continuity, there is a neighborhood around h where ORS is bounded away from zero. Hence a mean value theorem argument as in the last corollary, using the bound on $m\mathrm{ORS}(h)$ in the theorem, yields local Lipschitz continuity. If ORS is bounded away from zero a priori, then the same argument gives a uniform Lipschitz constant. ■

Remark 14 *Even for landscapes with small* ORS *values, ORS tends to be better-behaved than this corollary would indicate, but the result given is satisfactory for our purposes.*

While the previous theorem and its corollaries address the sensitivity of ORS to an arbitrary bounded change in the landscape, we get a sharper result if the change in the landscape occurs only far away from the reference point. This is important to the interpretation of ORS as a measure of local impressiveness, without regard to absolute elevation above the level of a distant ocean. As before, it is simpler to discuss ORS^2.

Theorem 15 (Locality) ORS^2 *is local: the contribution I to $\mathrm{ORS}^2(h)$ from points $\mathbf{x}$ with $\|\mathbf{x}\| > R$ satisfies*

$$I \leq \frac{2\,\|h\|_\infty^4}{\pi^2 R^2}.$$

Hence for every $h, k \in L^\infty$, if $h(\mathbf{x}) = k(\mathbf{x})$ for all $\mathbf{x}$ with $\|\mathbf{x}\| \leq R$, and $\|h\|_\infty, \|k\|_\infty \leq H$, then

$$\left|\mathrm{ORS}^2(h) - \mathrm{ORS}^2(k)\right| \leq \frac{2H^4}{\pi^2 R^2}$$

Proof. Let $E = \left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \geq R\right\}$. Then

$$\begin{aligned}
I &= \int_E f^2(h(\mathbf{x})/r)\; dA \\
&\leq \frac{2}{\pi^3} \int_E \left(\frac{h(\mathbf{x})}{r}\right)^4 dA \\
&= \frac{2}{\pi^3} \int_0^{2\pi} \int_R^\infty \left(\frac{h(\mathbf{x})}{r}\right)^4 r dr d\theta \\
&\leq \frac{4}{\pi^2} \int_R^\infty \frac{\|h\|_\infty^4}{r^3}\, dr \\
&\leq \frac{2\,\|h\|_\infty^4}{\pi^2 R^2}
\end{aligned}$$

■

We noted above that in the case of a low-slope cone, ORS is approximately quadratic in the height (for a fixed base radius) This is true in general as long as the slopes near the reference point are bounded.

Theorem 16 *For terrain that has bounded slope near the origin,* $\mathrm{ORS}(h)$ *approximately scales quadratically in the height (with no horizontal scaling). More precisely, assume that $h(\mathbf{x})/r$ is bounded and let $M > 0$. Then*

$$\mathrm{ORS}(Mh) = CM^2 + O(M^4)$$

as $M \to 0$, for some C depending on h.

Proof. Let $u(\mathbf{x}) = -h(\mathbf{x})/r$ and let $H = \|h\|_\infty$. Then the corresponding slope function for the (vertically) scaled landscape is Mu, and

$$\begin{aligned}
\mathrm{ORS}^2(Mh) &= \int_{\mathbb{R}^2} f^2(Mu(\mathbf{x}))\ dA \\
&= \int_{\mathbb{R}^2} \left[\frac{2}{\pi^3} M^4 u(\mathbf{x})^4 + g\left(Mu(\mathbf{x})\right)\right]\ dA \\
&= \frac{2}{\pi^3} M^4 \int_{\mathbb{R}^2} u(\mathbf{x})^4\ dA + \int_{\mathbb{R}^2} g\left(Mu(\mathbf{x})\right)\ dA
\end{aligned}$$

where $|g(u)| \leq C_1 u^6$ for all u. Hence

$$\begin{aligned}
\left|\mathrm{ORS}^2(Mh) - \frac{2}{\pi^3} M^4 \int_{\mathbb{R}^2} u(\mathbf{x})^4\ dA\right| &\leq \left|\int_{\mathbb{R}^2} g\left(Mu(\mathbf{x})\right)\ dA\right| \\
&\leq \int_{\mathbb{R}^2} \left|g\left(Mu(\mathbf{x})\right)\right|\ dA \\
&\leq \int_{\mathbb{R}^2} C_1 M^6 u(\mathbf{x})^6\ dA
\end{aligned}$$

Since h is bounded, u decays at least as $1/r$ at infinity, and it is assumed to be bounded at the origin. Hence

$$\int_{\mathbb{R}^2} u(\mathbf{x})^n\ dA < \infty \quad \text{for } n \geq 3$$

We can apply this for $n = 4$ to the expression above to see that

$$C_2 = \frac{2}{\pi^3} \int_{\mathbb{R}^2} u(\mathbf{x})^4\ dA$$

is finite. Applying the case $n = 6$ gives

$$\left|\mathrm{ORS}^2(Mh) - C_2 M^4\right| \leq C_3 M^6$$

where $C_3 = C_1 \int_{\mathbb{R}^2} u(\mathbf{x})^6\ dA$. Therefore

$$\mathrm{ORS}^2(Mh) = C_2 M^4 + O(M^6)$$

and

$$\mathrm{ORS}(Mh) = CM^2 + O(M^4)$$

as desired, where $C = \sqrt{C_2}$. ■

To state the next result, we return to considering ORS as a function of $\mathbf{p}$, h_0, and h. We look at how ORS depends on the horizontal location of the reference point, if we do not change its height. (This is a little strange physically, as the reference point is usually at ground level; we will address this immediately after the theorem.)

Theorem 17 *Let $H > 0$ be fixed. Then* ORS^2 *and ORS are continuous in* $\mathbf{p}$*, uniformly in* $\mathbf{p}$*,* h_0*, and* h*, provided that* $\|h_0 - h\|_\infty \leq H$*.*

We first need a lemma regarding $f^2\left(h/r\right)$.

Lemma 18 *Given r_1, r_2 with $0 < r_1 < r_2$, $f^2(h/r_1) - f^2(h/r_2)$ is an increasing function of h for $h \geq 0$.*

Proof. We have

$$\begin{aligned} \frac{d}{dh}\left(f^2\left(\frac{h}{r_1}\right) - f^2\left(\frac{h}{r_2}\right)\right) &= \left(f^2\right)'\left(\frac{h}{r_1}\right)\frac{1}{r_1} - \left(f^2\right)'\left(\frac{h}{r_2}\right)\frac{1}{r_2} \\ &> \frac{1}{r_1}\left(\left(f^2\right)'\left(\frac{h}{r_1}\right) - \left(f^2\right)'\left(\frac{h}{r_2}\right)\right) \\ &> 0 \end{aligned}$$

since f^2 is convex. ∎

Proof of the Theorem. We wish to bound $\left|\mathrm{ORS}^2\left(\mathbf{q}, h_0; h\right) - \mathrm{ORS}^2\left(\mathbf{p}, h_0; h\right)\right|$ for $\mathbf{q}$ near $\mathbf{p}$. Without loss of generality, we can let $\mathbf{p}$ be the origin, $h_0 = 0$, and $\mathbf{q} = (\delta, 0)$, for some $\delta > 0$, and we can look at the case where $\mathrm{ORS}^2\left(\mathbf{q}, h_0; h\right) \geq \mathrm{ORS}^2\left(\mathbf{p}, h_0; h\right)$. We have

$$\begin{aligned} \mathrm{ORS}^2\left(\mathbf{q}, 0; h\right) - \mathrm{ORS}^2\left(\mathbf{0}, 0; h\right) &\leq \int_{\mathbb{R}^2}\left(f^2\left(\frac{h(\mathbf{x})}{\|\mathbf{x}-\mathbf{q}\|}\right) - f^2\left(\frac{h(\mathbf{x})}{r}\right)\right) dA \\ &= \int_{\frac{\delta}{2}}^{\infty}\int_{-\infty}^{\infty}\left(f^2\left(\frac{h(\mathbf{x})}{\|\mathbf{x}-\mathbf{q}\|}\right) - f^2\left(\frac{h(\mathbf{x})}{r}\right)\right) dy\ dx \\ &\quad + \int_{-\infty}^{\frac{\delta}{2}}\int_{-\infty}^{\infty}\left(f^2\left(\frac{h(\mathbf{x})}{\|\mathbf{x}-\mathbf{q}\|}\right) - f^2\left(\frac{h(\mathbf{x})}{r}\right)\right) dy\ dx \\ &\leq \int_{\frac{\delta}{2}}^{\infty}\int_{-\infty}^{\infty}\left(f^2\left(\frac{h(\mathbf{x})}{\|\mathbf{x}-\mathbf{q}\|}\right) - f^2\left(\frac{h(\mathbf{x})}{r}\right)\right) dy\ dx \end{aligned}$$

where the second integral drops out because $\|\mathbf{x}-\mathbf{q}\| > r$ on that region and $f^2\left(h/r\right)$ is a decreasing function of r. Also, by the previous lemma, the difference between the f^2 values at a particular $\mathbf{x}$ will be maximized when $h(\mathbf{x})$ is as large as possible, so we have

$$\begin{aligned} \mathrm{ORS}^2\left(\delta, 0, 0; h\right) - \mathrm{ORS}^2\left(0, 0, 0; h\right) &\leq \int_{\frac{\delta}{2}}^{\infty}\int_{-\infty}^{\infty}\left(f^2\left(\frac{H}{\|\mathbf{x}-\mathbf{q}\|}\right) - f^2\left(\frac{H}{r}\right)\right) dy\ dx \\ &= \int_{-\frac{\delta}{2}}^{\infty}\int_{-\infty}^{\infty} f^2\left(\frac{H}{r}\right) dy\ du - \int_{\frac{\delta}{2}}^{\infty}\int_{-\infty}^{\infty} f^2\left(\frac{H}{r}\right) dy\ dx \\ &= \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}}\int_{-\infty}^{\infty} f^2\left(\frac{H}{r}\right) dy\ dx \end{aligned}$$

(where the first equality follows from the change of variables $u = x - \delta$), which is exactly ORS^2 applied to an infinitely long, thin "mesa" of constant height H.

This in turn can be estimated as follows, using Lemma 8:

$$
\begin{aligned}
\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\infty}^{\infty} f^2\left(\frac{H}{r}\right)\, dy\, dx \quad &= \quad \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} f^2\left(\frac{H}{r}\right)\, dy\, dx \\
&\quad +2\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{\frac{\delta}{2}}^{H} f^2\left(\frac{H}{r}\right)\, dy\, dx \\
&\quad +2\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \int_{H}^{\infty} f^2\left(\frac{H}{r}\right)\, dy\, dx \\
&\leq \quad 2\pi \int_{0}^{\frac{\sqrt{2}}{2}\delta} f^2\left(\frac{H}{r}\right)\, r\, dr \\
&\quad +2\delta \int_{\frac{\delta}{2}}^{H} f^2\left(\frac{H}{y}\right)\, dy \\
&\quad +2\delta \int_{H}^{\infty} f^2\left(\frac{H}{y}\right)\, dy \\
&\leq \quad 2\pi \cdot \frac{4}{\pi^2} H \cdot \frac{\sqrt{2}}{2}\delta \\
&\quad +2\delta \cdot \frac{4}{\pi} \cdot H \cdot \ln\left(\frac{2H}{\delta}\right) \\
&\quad +2\delta \cdot \frac{2}{\pi^3} \cdot \frac{H}{3}
\end{aligned}
$$

Hence we have

$$
\mathrm{ORS}^2\left(\delta, 0, 0; h\right) - \mathrm{ORS}^2\left(0, 0, 0; h\right) \to 0 \quad \text{as} \quad \delta \to 0 \quad (\text{with } H \text{ fixed})
$$

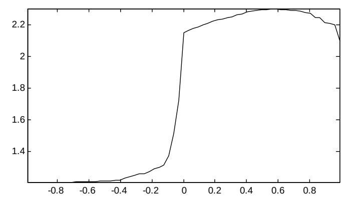
so ORS^2 and ORS are continuous in $\mathbf{p}$. Since the bound we derived only depends on H, and not on $\mathbf{p}$, h, or h_0, the continuity is uniform as desired. ■

(Note: if we put a bound on the slope of h near $\mathbf{p}$, this can be sharpened to yield Lipschitz continuity.)

We are usually interested in the case where $h_0 = h(\mathbf{p})$, yielding the function (with h fixed and $\mathbf{p}$ variable) $\mathrm{ORS}(\mathbf{p}, h(\mathbf{p}); h)$. Note that in general (when h is not continuous) we do not expect this function of $\mathbf{p}$ to be continuous, since the reference height follows the discontinuous function h. However, wherever h is continuous, Theorem 17 and Corollary 13 together imply that $\mathrm{ORS}(\mathbf{p}, h(\mathbf{p}); h)$ will also be continuous.

3 Examples

To get a feel for the meaning of ORS, it is most instructive to look at explicit examples, preferably with pictures. Below we display a sample cross-section for a few representative peaks. In addition, the website [5] and viewing packages



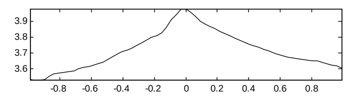


Table 2: El Capitan (left) and Mount Lyell (right) profiles

such as Google Earth[8] are very useful. All of the ORS values for the examples were generated by computer, using gridded digital elevation models (DEMs).[9]

First let us dispatch our introductory contrasting examples, Mount Elbert and Devils Thumb. Mount Elbert has an ORS of 237 meters, while Devils Thumb's is 828 meters, corresponding to their dramatically different profiles as shown in Table 1. These values show that a comparison between these two peaks based on ORS gives the opposite result from the comparison suggested by their absolute elevations.

Another illustrative contrast is provided by Yosemite National Park. The highest point in the park is Mount Lyell, at 3999 meters. It has a respectable ORS value of 200 meters. See Table 2. However, far more famous is the huge granite cliff on the side of Yosemite Valley known as El Capitan. It is hardly a mountain at all (there is higher terrain quite nearby), and its "summit" (a minor knoll some distance back from the brow of the cliff) has an elevation of only 2307 meters. El Capitan is a good example of a feature whose maximum ORS value is not obtained at the "summit" (local maximum of height). Rather, it is obtained by placing the reference point just atop the steepest portion of the cliff. The resulting ORS value is 575 meters. See Table 2. (The similarly famous and impressive Half Dome nearby gets an ORS of 580 meters; these are easily the two best ORS values in the park, and in the whole Sierra Nevada.) Here ORS clearly correlates much better with the notability of the features than does absolute elevation.

Table 3 lists the six U.S. states with the highest maximum ORS value. Not suprisingly, Alaska tops the list, although Mount McKinley (ORS = 1243 m, Elev = 6194 m) is not the best point in Alaska. The lower Mount Saint Elias is very close to tidewater (about 10 km away) , and is comparably steep, so it gets a higher ORS value. Most of the other peaks are well-known, except perhaps Mount Cleveland, the high point of Glacier National Park. (The glaciers there

[8] However note that, as of 2010, in some regions (typically non-U.S. regions with high relief), the dataset that underlies Google Earth is still of varying, and sometimes strikingly low, quality.

[9] The typical accuracy of the ORS values presented in this section is a few percent. More details on the calculations can be found on the peaklist website.[5]

Peak	ORS	Elev	State
Mount Saint Elias	1334	5489	Alaska
Mount Rainier	827	4392	Washington
Grand Teton	683	4197	Wyoming
Mount Shasta	675	4317	California
Mount Cleveland	672	3190	Montana
Mount Hood	649	3452	Oregon

Table 3: State best points by ORS

Peak	ORS	Elev	State
Nanga Parbat	1740	8125	Pakistan
Dhaulagiri	1680	8167	Nepal
Rakaposhi	1628	7788	Pakistan
Machhapuchhare	1596	6993	Nepal
Manaslu	1550	8163	Nepal

Table 4: World's top five independent peaks by ORS

are fast disappearing, but they have carved a number of exceptionally steep peaks.) It is interesting to also compare Mount Whitney, the high point of the contiguous U.S. (ORS = 418 m, Elev = 4421 m); note that it is bested within California not only by the huge stratovolcano Mount Shasta, but also by El Capitan and Half Dome (among others).

Worldwide, we have Table 4, which lists the top five independent[10] peaks in the world. Four are in the Himalaya, while Rakaposhi is in the nearby Karakoram range. While three of these peaks are in the famed group of fourteen "eight-thousanders" (with elevation over 8000 meters), two are not; in fact Machhapuchhare is not even in the top 300 peaks in the world by elevation. (It is a tremendously steep peak, near low terrain, in the Annapurna region of Nepal; it is highly sacred and is off-limits to climbing.) For comparison, Mount Everest, elevation 8848 m, gets a very respectable ORS value of 1302 m. Also note the dramatic difference in scale between these peaks and peaks in the contiguous U.S. (Mount Saint Elias does, however, come close to the top five, and actually beats Everest.)

4 Derived concepts

We have created two main concepts derived from ORS: *reduced ORS (RORS)* and *domain relief and steepness (DRS)*. We will discuss both briefly, without proofs.

[10] This list was actually generated by taking the five highest points as ranked by reduced ORS, as in Section 4, to ensure five truly independent peaks.

RORS is used for building a list of the "best" peaks (as judged in terms of relief and steepness) in a region. Since, for a fixed, continuous landscape function h, $\mathrm{ORS}(\mathbf{p}, h(\mathbf{p}); h)$ is a continuous function of $\mathbf{p}$, it is nonsensical to compile a list of points with the highest possible ORS values in a given region. This is true of height, as well; lists of the "highest N peaks" in a given region usually use some cutoff criterion to eliminate trivial subpeaks. Instead of pursuing this strategy, we created RORS, which is a variant of ORS which takes into account the degree of independence of a given peak from nearby "better" peaks.[11] Hence it measures a combination of relief, steepness, and independence. For details, we refer the reader to [5], but we can briefly note the most important feature of RORS. It is *automatically discrete*: for any $\varepsilon > 0$, the set of points $\mathbf{p}$ with $\mathrm{RORS}(\mathbf{p}) > \varepsilon$ is discrete (and hence finite, in a bounded domain). This makes it a valid list-making criterion; the list of the top N points in a given region, as ranked by RORS, is meaningful. Various such lists are presented on the website [5].

The second concept derived from ORS is more straightforward to define. It is a measure of the ruggedness of a given domain, taking into account both relief and steepness. It is easy to create such a measure using ORS: roughly, we (RMS) average the ORS value for every point in the domain, yielding what we call *domain relief and steepness*, DRS. However there are two additional issues. First, given a bounded domain $K \subset \mathbb{R}^2$, and a landscape function h, we redefine ORS to use sample points only within the given domain. Second, instead of declaring our modified slope integrand f to have $f(u) = 0$ for $u < 0$, we extend it as an even function.[12]

With notation as in Section 2, we define the new version of ORS, appropriate to this setting, as

$$\begin{aligned} \mathrm{ORS}(\mathbf{p}, h_0; h, K) &= \|f \circ u\|_{2,K} \\ &= \left[\iint\limits_K f^2\left(\frac{h_0 - h(\mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|}\right)\, dA(\mathbf{x}) \right]^{1/2} \end{aligned}$$

Then we define

$$\mathrm{DRS}(h, K) = \left[\frac{1}{A(K)} \iint\limits_K ORS^2(\mathbf{p}, h(\mathbf{p}); h, K)\ dA(\mathbf{p}) \right]^{1/2}$$

where $A(K)$ is the area of K. This can be expressed directly in terms of the (new) modified slope integrand f as follows. Abusing notation slightly, let

[11] Part of the inspiration for this strategy was *topographic prominence*, a popular alternate mountain measure. See for example [6].

[12] This change is not essential, but it does make the resulting formula more symmetric. It is easy to verify that using the original convention for f instead results in a definition of DRS that is $1/\sqrt{2}$ times that given here.

$u(\mathbf{p}, \mathbf{x}) = (h(\mathbf{p}) - h(\mathbf{x})) / \|\mathbf{p} - \mathbf{x}\|$. Then

$$\begin{aligned} \mathrm{DRS}(h, K) &= \frac{1}{\sqrt{A(K)}} \|f \circ u\|_2 \\ &= \left[\frac{1}{A(K)} \int\limits_{K \times K} f^2 \left(\frac{h(\mathbf{p}) - h(\mathbf{x})}{\|\mathbf{p} - \mathbf{x}\|} \right) \, dA(\mathbf{p}) \, dA(\mathbf{x}) \right]^{1/2} \end{aligned}$$

Note that this (quadruple) integral is symmetric in the variables $\mathbf{p}$ and $\mathbf{x}$, and that it has units of length, just as ORS does (recall that f is dimensionless).

We will not go into detail regarding DRS here; see [5] for more. However we will make two notes about it.

First, DRS is sensitive to the overall slope of the terrain, but it is continuous in the L^∞ norm, unlike a functional based on derivatives. Hence it will not give an unreasonably high value to a landscape with low relief, no matter how rugged, nor will its value depend (absurdly) on a particular microscale model of matter. (Think of applying the derivative to the surface of a "flat", "level" table, but taking into account the atomic-scale bumpiness of the surface—one will not obtain the expected value of zero.)

Second, empirical investigations indicate that the following problem is well-defined (with perhaps some mild regularity assumptions): within a given domain K_0, what is the domain $K \subset K_0$ with maximal ruggedness? Doing this is a tricky problem in calculus of variations, one which we have not investigated completely. However a coarse-gridded numerical approximation to this problem yields stable results. For example, our calculations indicate that the most rugged region in the contiguous 48 states is the Picket Range of the North Cascades, in Washington State.[5]

5 Acknowledgements

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